Pdf-03

(1) Algebra Dr. Satish Kr Gupta

State and prove first fundamental theorem on homomorphism of groups

Statement. Every homomorphic image of a group is isomorphic to some quotient group.

Proof. Let f be a homomorphism mapping from a group G to group G'

i.e. $f : G \longrightarrow G'$ be homomorphism

, $f(a) \in G'$, $\forall a \in G$

i.e $G' = \{f(a) \mid a \in G\}$

Let K be the Kernel of f

then $\frac{G}{K}$ is a quotient group

to prove $G' \cong \frac{G}{K}$

Define a mapping

$\phi : \frac{G}{K} \longrightarrow G'$ such that

$\phi(Ka) = fa, \ \forall a \in G$

Clearly ϕ is well defined

To show ϕ is one-one Let $Ka, Kb \in \frac{G}{K}$, $a, b \in G$

$\phi(Ka) = \phi(Kb) \Rightarrow f(a) = f(b)$

$\Rightarrow f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$

$\Rightarrow f(a)[f(b)]^{-1} = e'$

$\Rightarrow f(ab^{-1}) = e'$

$\Rightarrow ab^{-1} \in K$

$\Rightarrow Ka = Kb$ $[\because Ha = Hb \Leftrightarrow ab^{-1} \in H]$

ϕ is onto. Let $f(a) \in G' \Rightarrow \exists Ka \in \frac{G}{K}$

such that $\phi(Ka) = fa, \forall a \in G$

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ϕ preserves group compositions,

$\phi[(Ka)(Kb)] = \phi[Kab]$ $[\because K \Delta G]$

$= f_{ab}$

$= f_a f_b$ $[\because f$ is homomorphism$]$

$= \phi(Ka)\phi(Kb)$

Hence $\frac{G}{K} \cong G'$, proved.

Note. (1) G' = homomorphism image of G

K = Kernel of homomorphism on G

$\boxed{\frac{G}{K} \cong \text{homomorphic image of } G}$

(2) Set of all right cosets of Kernel of homomorphism of G is isomorphic to homomorphic image of G

i.e $\{Ka \mid a \in G\} = G' = \{f(a) \mid a \in G\}$

, K is Kernel of homomorphism on G.

Ex. If n be any given positive integer, show that the mapping $f: C_0 \rightarrow C_0$ defined by $f(z) = z^n$ is an endomorphism of the multiplicative group of non-zero complex numbers. What is the kernel of this endomorphism.

Soln. Let C_0 be a multiplicative group.

Let $f: C_0 \rightarrow C_0$ such that $f(z) = z^n$

To prove f is endomorphism

$z_1 \in C_0 \Rightarrow f(z_1) = z_1^n$

$z_2 \in C_0 \Rightarrow f(z_2) = z_2^n$

$f(z_1 z_2) = (z_1 z_2)^n = z_1^n \cdot z_2^n = f(z_1) f(z_2)$

$\Rightarrow f$ is an endomorphism of C_0

Let K = Kernel of f

then $K = \{z \in C_0 \mid f(z) = 1$, identity element of $C_0\}$

$K = \{z \in C_0 \mid z^n = 1 \Rightarrow z = (1)^{1/n}$, i.e n^{th} root of unity$\}$

$K = \{(1)^{1/n} \mid n \in N\}$

$(1)^{1/n}$ are $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$, $\omega = \text{cis}\,\frac{2\pi}{n}$

Q. Define automorphism of a group.

Ans. Let G be a group

then an isomorphism mapping $f: G \to G$

is called an automorphism.

ex. 1 let $(I, +)$ be a group.

then $f: I \to I$ such that $f(x) = -x, \forall x \in I$

is an automorphism

Proof. <u>f is one-one</u>. let $x, y \in$ Domain of f.

$f(x) = f(y) \Rightarrow -x = -y$
$\Rightarrow x = y$

<u>f is onto</u>. let $y \in$ range of $f \Rightarrow \exists$ $(-y) \in$ domain of f

such that $f(-y) = -(-y) = y$,

Finally let $x, y \in$ Domain of f then

$\cancel{f(xy)}\ f(x+y) = -(x+y)$ [$* \to +$, here]

$= (-x) - y$

$= (-x) + (-y)$

$= f(x) + f(y)$

$\therefore$ f is an authomorphism

2 Show $a \to a^{-1}$ is an ~~autho~~ automorphism of a group G

iff G is abelian

Solution. let $f: G \to G$ such that

$f(x) = x^{-1}$, $\forall x \in$ domain of f.

<u>"if part"</u> suppose G is <u>abelian</u>, To prove f is automorphism

<u>f is one-one</u>.

$f(x) = f(y) \Rightarrow x^{-1} = y^{-1}$

$\Rightarrow (x^{-1})^{-1} = (y^{-1})^{-1} = x = y$

<u>f is onto</u> let $y \in$ Range of $f \Rightarrow \exists\ y^{-1} \in$ domain of f

such that $f(y^{-1}) = (y^{-1})^{-1} = y$.

Finally. Let $x, y \in$ domain of f

$$f(xy) = (xy)^{-1} \quad [\because (ab)^{-1} = b^{-1}a^{-1} \text{ in a group}]$$
$$= y^{-1}x^{-1}$$
$$= x^{-1}y^{-1} \quad [\because G \text{ is abelian}]$$
$$= f(x)f(y).$$

Only if part: Suppose f is automorphism

To prove G is abelian

Let $a, b \in G \Rightarrow ab \in G$

Now $f(ab) = (ab)^{-1} = b^{-1}a^{-1} = f(b)f(a)$

$f(ab) = f(ba)$ [f is an automorphism]

$\Rightarrow ab = ba$ [$\because f$ is one-one]

$\Rightarrow G$ is abelian.

(3) Let f be an automorphism on a group G.

Let H be any subgroup of G. Prove $f(H)$ is a subgroup of G

Soln. Let $f: G \to G$ be an automorphism

Let H be a subgroup of G

then $f(H) = \{x' \in \text{range of } f \text{ s.t. } x' = f(h), h \in H\}$

$f(H) = \{f(h) \mid h \in H\}$

Let ~~a, b~~ $a = f(h_1)$, $h_1 \in H$

and $b = f(h_2)$, $h_2 \in H$

To show $ab^{-1} \in f(H)$

We have $h_1 h_2^{-1} \in H$, H is a subgroup (it is given $h_1 \in H, h_2 \in H$)

$$f(h_1 h_2^{-1}) = f(h_1) f(h_2^{-1}) \quad [f \text{ is automorphism}]$$
$$= f(h_1)[f(h_2)]^{-1}$$
$$= ab^{-1} \in f(H)$$

$\Rightarrow f(H)$ is a subgroup of G.

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Theorem. Let $A(G)$ = set of all automorphism on a group

i.e $A(G) = \{ f : f : G \to G$ is an authomorphism$\}$

To prove $A(G)$ forms a group under composite of fns as composition.

(1) $A(G)$ is closed. Let $f, g \in A(G)$

then

$$gf(ab) = g[f(ab)]$$
$$= g[f(a)f(b)] \quad [\because f \text{ is an automorphism}]$$
$$= gf(a) \cdot gf(b) \quad [\because g \text{ is an automorphism}$$
$$= [gf(a)][gf(b)]$$

$\Rightarrow gf \in A(G) \Rightarrow A(G)$ is closed.

(2) Associativity. We know that composite mapping is always associative

i.e $(fg)h = f(gh), \ \forall f, g, h \in A(G)$

(3) Existance of identy element.

i is the identity function in $A(G)$

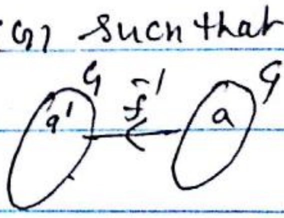
such that
$$i(ab) = ab$$
$$= i(a)\, i(b)$$

Thus if $f \in A(G)$, $i \in A(G)$ then
$$if = f = fi.$$

(4) Existance of inverse. Let $i \in A(G)$, identity functio

$f \in A(G)$ then $\exists f^{-1} \in A(G)$ such that

$$ff^{-1} = i$$



Let $a \in G \Rightarrow \exists a' \in G$ s.t

$f^{-1}(a) = a' \Rightarrow f(a') = a$

Similarly $f^{-1}(b) = b' \Rightarrow f(b') = b$

$$f^{-1}(ab) = f^{-1}[f(a')f(b')] = f^{-1}[f(a'b')] = a'b'$$
$$= f^{-1}(a)\, f^{-1}(b). \Rightarrow f^{-1} \text{ is an automorphism}$$
$$\Rightarrow f^{-1} \in A(G).$$

$\Rightarrow A(G)$ is a group.

Inner automorphism. Let G be a group
$a \in G$
then a mapping
$f_a : G \longrightarrow G$ ~~is called~~ such that
$f_a(x) = a^{-1} x a, \ \forall \ x \in G$ is called
an inner automorphism provided f_a is an authomorphism

$y = a^{-1} x a$
$a y a^{-1} = a(a^{-1} x a) a^{-1}$

Proof. f_a is one-one
$f_a(x) = f_a(y) \Rightarrow a^{-1} x a = a^{-1} y a$
$\Rightarrow xa = ya$ [left cancellation law]
$\Rightarrow x = y$ [by right cancellation law]

f_a is onto let $y \in$ range of f then $\exists \ (a y a^{-1}) \in$ domain of f_a
such that $f_a[a y a^{-1}] = a^{-1}(a y a^{-1}) a$
$= (a^{-1} a) y (a^{-1} a) = e y e = y.$

Finally,
$$f_a(xy) = a^{-1}(xy) a$$
$$= a^{-1} x a a^{-1} y a \qquad [xy = xey = x a a^{-1} y.]$$
$$= (a^{-1} x a)(a^{-1} y a)$$
$$= f_a(x) f_a(y)$$
$\Rightarrow f_a$ is an automorphism.

Theorem. The set $I(G)$ of all inner automorphisms of a group G is a normal subgroup of the group of its automorphisms isomorphic to the quotient group $\frac{G}{Z}$ of G where Z is the centre of G.

Proof.

Let $A(G)$ be the group of all automorphisms of G. $[A(G) = \{f : f : G \to G$ is isomorphisms$\}]$

Then $I(G) \subseteq A(G)$ $[I(G) = \{f_a : f_a : G \to G$, $f_a(x) = a^{-1}xa, \forall x \in G$, $a \in G$, fixed$\}]$

(i) To show $f_{a^{-1}} = f_a^{-1}$ $[\text{i.e } (f_a)^{-1} = f_{a^{-1}}]$

Let $x \in G$ then

$$(f_a f_{a^{-1}})(x) = f_a[f_{a^{-1}}(x)] = f_a[(a^{-1})^{-1}x(a^{-1})]$$
$$= f_a[axa^{-1}] = a^{-1}(axa^{-1})a = exe = x$$
$$= I(x)$$

$\Rightarrow (f_a)^{-1} = f_{a^{-1}}$.

(ii) To show $f_a f_b = f_{ba}$

Let $x \in G$

$$(f_a f_b)(x) = f_a[f_b(x)] = f_a[b^{-1}xb]$$
$$= a^{-1}(b^{-1}xb)a = (ba)^{-1}x(ba) = f_{ba}(x)$$

$\Rightarrow f_a f_b = f_{ba}$, $\forall a, b \in G$.

To prove $I(G)$ is a subgroup of $A(G)$.

Let $f_a \in I(G)$, $f_b \in I(G)$

To show $f_a(f_b)^{-1} \in I(G)$

We have $f_a(f_b)^{-1} = f_a f_{b^{-1}} = f_{b^{-1}a} \in I(G)$ as $b^{-1}a \in G$

Thus $f_a \in I(G), f_b \in I(G) \Rightarrow f_a(f_b)^{-1} \in I(G)$

$\Rightarrow I(G)$ is a subgroup of $A(G)$.

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To prove $I(G) \triangleleft A(G)$

Let $f \in A(G)$, $f_a \in I(G)$, $a \in G$

$$f f_a f^{-1}(x) = f[f_a\{f^{-1}(x)\}]$$
$$= f[a^{-1}\{f^{-1}(x)\}a]$$
$$= f(a^{-1})\, f[f^{-1}(x)]\, f(a)$$
$$= f(a^{-1})\, x\, f(a)$$
$$= [f(a)]^{-1} x\, f(a)$$
$$= c^{-1} x c, \quad f(a) = c, \text{ as } f(a) \in G \;\because f: G \to G,$$
$$= f_c(x)$$
$$\Rightarrow f f_a f^{-1} \in I(G)$$
$\therefore$ $I(G) \triangleleft A(G)$.

Let Z be the centre of G

then $Z = \{z \in G \mid zx = xz, \forall x \in G\}$

To prove $\frac{G}{Z} \cong I(G)$

Define a mapping
$$\phi : G \longrightarrow I(G) \text{ such that}$$
$$\phi(a) = f_{a^{-1}}, \forall a \in G$$

<u>ϕ is onto</u>

Let $f_a \in I(G) \Rightarrow \exists\, a^{-1} \in G$ such that
$$\phi(a^{-1}) = f_{(a^{-1})^{-1}} = f_a.$$

To show
$\phi(ab) = \phi(a)\phi(b)$

~~LHS =~~ Next page

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To prove $\phi(ab) = \phi(a)\phi(b)$

$$\begin{aligned} LHS &= \phi(ab) \\ &= f_{(ab)^{-1}} \\ &= f_{b^{-1}a^{-1}} \\ &= f_{a^{-1}} f_{b^{-1}} \quad [f_a f_b = f_{ba}] \\ &= \phi(a)\phi(b). \end{aligned}$$

$\Rightarrow \phi$ is homomorphism onto.

Let K be the Kernel of ϕ

To prove $K = Z$

Let $z \in K \Leftrightarrow \phi(z) =$ identity function of $I(G)$

$$\begin{aligned} &\Leftrightarrow f_{z^{-1}} = I \\ &\Leftrightarrow f_{z^{-1}}(x) = I(x), \ \forall x \in G \\ &\Leftrightarrow (z^{-1})^{-1} x z^{-1} = x, \ \forall x \in G \\ &\Leftrightarrow z x z^{-1} = x, \ \forall x \in G \\ &\Leftrightarrow z x z^{-1} z = x z, \ \forall x \in G \\ &\Leftrightarrow z x = x z, \ \forall x \in G \end{aligned}$$

$z \in K \Leftrightarrow z \in Z$

$\Rightarrow K = Z$

Hence By first fundamental theorem on homomorphism on groups

$$I(G) \cong \frac{G}{\text{Kernel of } \phi}$$

$$\Rightarrow I(G) \cong \frac{G}{Z}$$

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Theorem. Let G be a group. Let H be a normal subgroup of G. If K is a normal subgroup of G containing H then the quotient group $\frac{K}{H}$ is a normal subgroup of the quotient group $\frac{G}{H}$. Conversely, if $\frac{K}{H}$ is a normal subgroup of $\frac{G}{H}$, then K is a normal subgroup of G containing H.

Proof. Let H be a normal subgroup of a group G

Let K be a normal subgroup of G such that

$H \subseteq K$.

then H is a normal subgroup of K $\left[\begin{array}{c} \underset{\in G}{k}\, h\, \underset{\in G}{k^{-1}} \in H \text{ as } H \triangleleft G \\ K \subseteq G \end{array}\right]$

$\Rightarrow \frac{K}{H}$ is a quotient group.

Now, $\frac{K}{H} = \{ Ha \mid a \in K \}$

Let $Ha \in \frac{K}{H} \Rightarrow Ha \in \frac{G}{H}$ $\left[\because a \in K \Rightarrow a \in G \text{ as } K \subseteq G\right]$

$\Rightarrow \frac{K}{H} \subseteq \frac{G}{H}$

$\Rightarrow \frac{K}{H}$ is a subgroup of $\frac{G}{H}$

To show $\frac{K}{H} \triangleleft \frac{G}{H}$

Let $Hk \in \frac{K}{H}$, $Hg \in \frac{G}{H}$

$Hg\,Hk\,(Hg)^{-1} = Hg\,Hk\,Hg^{-1}$ $\left[\because (Hg)^{-1} = Hg^{-1},\ H \triangleleft G\right]$

$= Hgkg^{-1}$ $[H \triangleleft G]$

$\Rightarrow Hgkg^{-1} \in \frac{K}{H}$ as $gkg^{-1} \in K$ as $K \triangleleft G$

$\Rightarrow Hg\,Hk\,(Hg)^{-1} \in \frac{K}{H}$ $\left[\because Hg\,Hk\,(Hg)^{-1} = Hgkg^{-1}\right]$

<u>Converse:</u> If $\frac{K}{H} \triangleleft \frac{G}{H}$

To prove $K \triangleleft G$ such that $H \subseteq K$.

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Given

$$\frac{K}{H} \triangleleft \frac{G}{H}$$

$\Rightarrow Hx\,Hk\,(Hx)^{-1} \in \frac{K}{H}$ $\left[\text{if } Hx \in \frac{G}{H},\ Hk \in \frac{K}{H}\right]$

$\Rightarrow Hxkx^{-1} \in \frac{K}{H}$

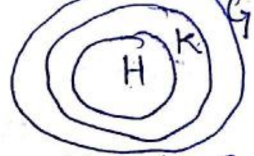
$\Rightarrow xkx^{-1} \in K$

$\Rightarrow K \triangleleft G$ as $x \in G$

Also $\frac{K}{H}$ is a quotient group $\Rightarrow H \triangleleft K$

Now $H \triangleleft K$ and $K \triangleleft G$

$\Rightarrow H \subseteq K$



$\therefore$ $K \triangleleft G$ such that K contains H. $H \subseteq K \subseteq G$,

proved.

Theorem. State and prove second law of isomorphism on groups.

Statement. Let H be a normal subgroup of a group G and K be a normal subgroup of G such that $H \subseteq K$, then $\frac{G}{K} \cong \frac{G/H}{K/H}$.

Proof. Let H and K be any two normal subgroups of a group G such that $H \subseteq K \subseteq G$

Then To show $\frac{G}{K} \cong \dfrac{\frac{G}{H}}{\frac{K}{H}}$

We know that if H and K be two normal subgroups of G such that $H \subseteq K$

$\Rightarrow$ H is a normal subgroup of K $\left[\because\ \underset{\in G}{k}\,\underset{\in G}{h}\,k^{-1} \in H,\ H \triangleleft G,\ K \subseteq G.\right]$

$\Rightarrow \frac{K}{H}$ is a quotient group ——— (1)

Now, $\frac{K}{H} = \{ Hk \mid k \in K \}$

Also we have ——— (2)

$\frac{G}{H} = \{ Hg \mid g \in G \}$

(1) & (2) $\Rightarrow \frac{K}{H}$ is a subgroup of $\frac{G}{H}$ $\left[(Hk_1)(Hk_2)^{-1} \in \frac{K}{H}\right]$ as $k_1 k_2^{-1} \in K$

To show $\frac{K}{H} \triangleleft \frac{G}{H}$

~~Let~~ Consider

$$Hg\,Hk\,Hg^{-1} = Hgkg^{-1} \quad [\because H \triangleleft G]$$

$$\Rightarrow Hgkg^{-1} \in \frac{K}{H} \quad [\because gkg^{-1} \in K \text{ as } K \triangleleft G]$$

Define a mapping

$\phi : \frac{G}{H} \longrightarrow \frac{G}{K}$ such that

$$\phi(Hg) = Kg, \ \forall\, g \in G$$

Clearly ϕ is well defined as $\left[\because Ha = Hb \Leftrightarrow ab^{-1} \in H \text{ Here } g_1, g_2 \in G\right]$

$$Hg_1 = Hg_2 \Rightarrow g_1 g_2^{-1} \in H$$

$$\Rightarrow g_1 g_2^{-1} \in K \quad [H \subseteq K]$$

$$\Rightarrow Kg_1 = Kg_2$$

$$= \phi(Hg_1) = \phi(Hg_2).$$

To show ϕ is homomorphism let $Hg_1, Hg_2 \in \frac{G}{H}$, $g_1, g_2 \in G$

$$\phi[(Hg_1)(Hg_2)] = \phi[Hg_1 g_2] \quad [\because HaHb = Hab \text{ as } H \triangleleft G]$$

$$= Kg_1 g_2$$

$$= Kg_1\, Kg_2 \quad [\because K \triangleleft G]$$

$$= \phi(Hg_1)\,\phi(Hg_2)$$

$\Rightarrow \frac{G}{K}$ is homomorphic image of $\frac{G}{H}$. — (3)

To find Kernel of ϕ

Identity element in $\frac{G}{K}$ is K.

We know Kernel of ϕ = set of all those elements which are mapped on identity element of $\frac{G}{K}$.

Let $Hg \in$ Kernel of $\phi \Leftrightarrow \phi(Hg) = K$

$$\Leftrightarrow Kg = K$$

$$\Leftrightarrow g \in K \quad [\because g \in K \Rightarrow Hg = K]$$

$$\Leftrightarrow Hg \in \frac{K}{H}$$

so Kernel of $\phi = \frac{K}{H}$ — (4)

By 1st fundamental theorem of homomorphism on groups, we have

$$\frac{\frac{G}{H}}{\text{Kernel of } \phi} \cong \text{homomorphic image of } \frac{G}{H}$$

$$\Rightarrow \quad \frac{\frac{G}{H}}{\frac{K}{H}} \cong \frac{G}{K} \quad [\text{from (3) \& (4)}]$$

, proved.

Theorem. State and prove 3rd law of Isomorphism on groups.

Statement. Let H be any subgroup of a group G

Let N be any normal subgroup of G,

then $\frac{HN}{N} \cong \frac{H}{H \cap N}$

Proof. Suppose H be any subgroup of a group G

Let N be any normal subgroup of G

$\Rightarrow HN$ be a subgroup of G [$\because$ H is a subgroup of G, N is a subgroup of G $\Rightarrow HN$ is a subgroup $\Leftrightarrow HN = NH$]

Now, $HN = \{hn \mid h \in H, n \in N\}$

$N = \{n \mid n \in N\}$

clearly $N \subseteq HN$ [$\because$ $hn \in HN$ then $hn = n$ if $h = e$

To prove $N \triangleright HN$ $\quad \therefore$ $en \in N \Rightarrow en \in HN$ $\Rightarrow N \subseteq HN$]

let $n \in N$, $h_1 n_1 \in HN$

then $h_1 n_1 n (h_1 n_1)^{-1} = h_1 \underline{n_1 n n_1^{-1}} h_1^{-1} \in N$ [$\because$ $n_1 n n_1^{-1} \in N$ as $N \triangleleft G$, $h_1 \in G$]

$\Rightarrow N \triangleright HN$

$\Rightarrow \frac{HN}{N}$ is a quotient group — (1)

We know

$H \cap N$ is a subgroup of G

[$\because$ Intersection of two subgroups is a subgroups]

clearly

$(H \cap N) \subseteq H \subseteq G$

$\Rightarrow H \cap N$ is a subgroup of H

Also $(H \cap N)$ will be normal subgroup of H

[$n \in H \cap N \Rightarrow n \in H, n \in N$
(i) $n \in H$ then $\cancel{hnh}$ $hnh^{-1} \in H$
(ii) $n \in N$, $hnh^{-1} \in N$, $h \in G$, $N \triangleleft G$
$\Rightarrow hnh^{-1} \in H \cap N$]

$\Rightarrow \dfrac{H}{H \cap N}$ is a quotient group — (2)

To show $\dfrac{HN}{N} \cong \dfrac{H}{H \cap N}$

Define a mapping

$\phi : H \longrightarrow \dfrac{HN}{N}$ such that

$\phi(h) = Nh, \ \forall h \in H$

[$\because H \subseteq HN$
$\Rightarrow h \in H \Rightarrow h \in HN$]

clearly ϕ is well defined

<u>To show ϕ is onto</u>

Let $Nh \in \dfrac{HN}{N} \Rightarrow \exists h \in H$ such that $\phi(h) = Nh$, $\forall h \in H$.

Again Let $x, y \in H$

$\phi(xy) = Nxy$ [$\because N \triangleleft HN$]

$\phi(xy) = Nx\,Ny$ $\Rightarrow N \triangleleft H$ as $H \subseteq HN \subseteq G$]

$\phi(x,y) = \phi(x)\,\phi(y)$

$\Rightarrow \phi$ is homomorphism

$\therefore \dfrac{HN}{N}$ is a homomorphic image of H — (3)

<u>To find kernel of ϕ</u> Let $h \in$ Kernel of ϕ

$\Rightarrow \phi(h) =$ Indentity element of $\dfrac{HN}{N}$

$\Leftrightarrow \phi(h) = N$

$\Leftrightarrow Nh = N$

$\Leftrightarrow h \in N$

$\Leftrightarrow h \in H \cap N$ [$\because h \in H, h \in N$
$\Rightarrow h \in H \cap N$]

$\Rightarrow$ Kernel of $\phi = (H \cap N)$ — (4)

So using first fundamental theorem of homo. on groups

$\dfrac{HN}{N} \cong \dfrac{H}{H \cap N}$ [$\because$ Homomorphic Image of $H \cong \dfrac{H}{\text{Kernel of } \phi}$
as $\phi : H \to \dfrac{HN}{N}$]